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AN EMPIRICAL BAYES APPROACH TO DECISIONS BASED ON MULTIVARIATE TEST DATA

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ABSTRACT

A general framework for making mastery/nonmastery decisions based on multivariate test data is described in this study. Over all, mastery is granted (or denied) if the posterior expected loss associated with such action is smaller than the one incurred by the denial (or grant) of mastery. An explicit form for the cutting contour which separates mastery and nonmastery states in the test score space is given for multivariate test scores which follow a normal distribution with a constant loss ratio. For the case involving multiple cutting scores in the true ability space, the test score cutting contour will resemble the boundary defined by multiple test cutting scores when the test reliabilities are reasonably close to unity. For tests with low reliabilities, decisions may very well be based simply on a suitably chosen composite score.

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1. INTRODUCTION

Application of mental measurement to selection or certification problems often involves the use of more than one test score. For

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example, the selection of students for an advanced program in some subject area may be based on several traits (variables), such as prior achievement, aptitude, interest, etc. Ideally, selection should be based on the subject's true measures on these traits; in reality, however, decisions are typically based on observed test scores which are contaminated with errors of measurement. Thus, misclassifications are bound to occur, and rules for decisions based on test data are typically formulated in such a way as to minimize the risks incurred by misclassification.

Decision problems based on one variable have been considered at length in the literature. Statistical issues involved in establishing a single cutoff (cutting, passing, or mastery) score are described in detail in a number of sources including Swaminathan, Hambleton, and Algina (1975); Huynh (1976, 1977, 1979, 1980); Wilcox (1976); and van der Linden and Mellenberg (1977). Huynh (1979, 1980) also provides an explicit relationship among test cutting score, losses incurred by misclassification, and errors of measurement. In general, within the minimax or empirical Bayes decision framework, it is found that errors in measurement will reduce the test cutting score when a false negative error is more serious than a false positive error. Conversely, the test cutting score will increase when a false negative error is less serious than a false positive error.

The effect of errors of measurement in selection situations involving multiple true cutting scores has been considered by Lord (1962). The selection framework used involves the regression line expressing the amount of "desirability" assigned to different examinees as a function of the observed test scores. Using the multivariate normal distribution to describe the true and observed scores, Lord was able to plot the contour line in the observed test score plane which separates the subjects deemed acceptable (masters) from those judged as unacceptable (nonmasters). Lord's paper, however, does not appear to come naturally from decision theory as formulated by Wald (1950) or as prescribed in Ferguson (1967).

The purpose of this paper is twofold. First it will describe a general empirical Bayes solution to the "plotting" of a cutting contour in selection situations involving multiple test scores. Second, it will explore the influence of the loss ratio on the cutting contour and will reexamine the distortion caused by errors of measurement (Lord, 1962), using an empirical Bayes decision-theoretic framework. Examples based on the multivariate normal distribution with constant losses for misdecisions are provided to illuminate various points or procedures put forward in the paper.

2. EMPIRICAL BAYES APPROACH TO CUTTING CONTOUR

Now let the vector $\theta = (\theta_1, \theta_2, \ldots, \theta_k)'$ denote the true scores (measures) of an individual subject on k traits (or selection variables). Let Ω represent the region in the true score space where a subject must be located in order to qualify for the true state of mastery. Thus a subject is defined as a true master if $\theta \in \Omega$. Let Ω^c be the complement of Ω. Then a subject is declared a true nonmaster when $\theta \in \Omega^c$.

Now let the vector $x = (x_1, x_2, \ldots, x_k)'$ represent the observed test scores of the subject. On the basis of x and other prior information regarding θ, a decision may be made concerning the subject: either to grant mastery (action a_1) or to deny mastery (action a_2). When $\theta \in \Omega$, the best course of action is a_1, and no loss will be encountered. Similarly, action a_2 is best when $\theta \in \Omega^c$. For other situations, classification errors occur. To be specific, the choice of action a_2 when $\theta \in \Omega$ constitutes a false negative error, whereas the selection of a_1 when $\theta \in \Omega^c$ produces a false positive error.

Let $C_s(\theta)$ be the loss associated with a false negative error and $C_f(\theta)$ be the loss encountered by a false positive error. Let $p(\theta|x)$ be the posterior probability density of θ given that the test score vector x has been observed. Given x, the posterior expected loss encountered in taking action a_1 is given by the integral $R(a_1|x) = \int_{\Omega^c} C_f(\theta)p(\theta|x)d\theta$.

Similarly, the posterior loss associated with the choice of action a_2 is $R(a_2|x) = \int_\Omega C_s(\theta)p(\theta|x)d\theta$.

It follows from Bayes (or empirical Bayes) decision theory as expressed, for example, in Ferguson (1967) that, in the test score space generated by the test score vector x, the cutting contour S separating the two actions a_1 (granting mastery) and a_2 (denying mastery) is defined by the equality $R(a_1|x) = R(a_2|x)$. In other words, the line (or surface) S consists of all points x at which

$$\int_\Omega C_s(\theta)p(\theta|x)d\theta = \int_{\Omega^c} C_f(\theta)p(\theta|x)d\theta. \quad (1)$$

The following section explores in detail the implications of Equation (1) for the case involving constant losses and multiple true cutting scores.

3. CUTTING CONTOUR FOR CONSTANT LOSSES AND MULTIPLE TRUE CUTTING SCORES

Let losses be constant and expressed as $C_f(\theta) = 1$ and $C_s(\theta) = Q$ in the region where they do not vanish. In other words, Q is the ratio of the false negative loss to the false positive loss. In addition, let Ω be the "upper right" corner defined by the true cutting scores $\theta_1^*, \theta_2^*, \ldots, \theta_k^*$. In other words,

$$\Omega = \{\theta; \theta_1^* \leq \theta_1, \theta_2^* \leq \theta_2, \ldots, \theta_k^* \leq \theta_k\}.$$

With constant losses Equation (1) may now be written as

$$Q\int_\Omega p(\theta|x)d\theta = \int_{\Omega^c} p(\theta|x)d\theta.$$

Since $\Omega \cup \Omega^c$ spans the entire space for θ, it follows that

$$\int_\Omega p(\theta|x)d\theta + \int_{\Omega^c} p(\theta|x)d\theta = 1.$$

With this relationship, Equation (1) becomes

$$\int_\Omega p(\theta|x)d\theta = \frac{1}{1+Q}, \quad (2)$$

which may be written, using the given multiple true cutting scores, as

$$Pr(\theta_1^* \leq \theta_1, \theta_2^* \leq \theta_2, \ldots, \theta_k^* \leq \theta_k|x) = \frac{1}{1+Q} \quad (3)$$

The line consisting of the points of coordinate x which satisfy Equation (2) or (3) defines the boundary between granting and denying mastery in the test score space. This boundary line will be referred to as a cutting contour.

4. CUTTING CONTOUR IN MULTIVARIATE NORMAL TEST SCORES

For illustrative purposes, let it be assumed that the true score vector θ for a population of subjects follows a multivariate normal distribution with mean vector $\mu = (\mu_1, \mu_2, \ldots, \mu_k)'$ and with covariance matrix $\Sigma_\theta = (\sigma_{ij})$. In the terminology of empirical Bayes statistics, this statement is equivalent to the requirement that the prior distribution of the true score vector θ be the same for all subjects in the population under study. This common prior distribution may be estimated from historical test score data or by procedures which are consistent with classical measurement theory and practice.

The difference vector $e = x - \theta$ represents the errors of measurement. It will be assumed that the k components of e are normally and independently distributed, each with a mean of zero and a variance of ε_{ii}, $i = 1,2,\ldots,k$, free of θ. In addition, it will be assumed that the two vectors e and θ are stochastically independent. To simplify the notation, let Σ_e be the diagonal matrix with elements ε_{ii}.

It follows from classical measurement theory and from known properties of multivariate normal distributions that the joint distribution of x and θ is multivariate normal with a mean vector of μ for both x and θ and with a covariance matrix defined as

$$\left(\begin{array}{c|c} \Sigma_x & \Sigma_\theta \\ \hline \Sigma_\theta & \Sigma_\theta \end{array}\right)$$

where $\Sigma_x = \Sigma_\theta + \Sigma_\varepsilon$. Hence the posterior distribution of θ given the test score x is multivariate normal with mean vector $\xi(x) = (\xi_1, \xi_2, \ldots, \xi_k)' = \mu + (x-\mu)'\Sigma_\theta\Sigma_x^{-1}$ and with covariance matrix $\Lambda = (\lambda_{ij}) = \Sigma_\theta - \Sigma_\theta\Sigma_x^{-1}\Sigma_\theta$. The vector $\xi(x)$ is a function of the test score vector x. On the other hand, the matrix Λ is free of x.

Now let us consider the standardized variables $y_1, y_2, \ldots, y_k$ where

$$y_i = (\theta_i - \xi_i(x))/\sqrt{\lambda_{ii}}, \quad i = 1,2,\ldots,k.$$

Each of these variables has zero mean and unit variance. Let Γ be the correlation matrix associated with Λ (i.e., Γ is the covariance matrix of the y_i variables). In addition, let

$$y_i^* = (\theta_i^* - \xi_i(x))/\sqrt{\lambda_{ii}}, \quad i = 1,2,\ldots,k. \tag{4}$$

Then the cutting contour separating the two actions a_1 and a_2 in the test score space is defined by the equality

$$\Pr(y_1^* \leq y_1, y_2^* \leq y_2, \ldots, y_k^* \leq y_k) = \frac{1}{1+Q} \tag{5}$$

where the random vector $y = (y_1, y_2, \ldots, y_k)'$ follows a multivariate normal distribution with zero means, unit variances, and correlation matrix Γ free of x.

Consider now the set γ consisting of the points with coordinates $(y_1^*, y_2^*, \ldots, y_k^*)$ which satisfy Equation (5). Tihansky (1970) refers to this set as an equidistributional contour and provides ways to construct contours of this type for bivariate normal distributions. The contour γ depends only on Γ which does not involve the observed test score vector x. Once it has been constructed, the cutting contour C in the test score space may be plotted via the system of linear equations represented by

$$\mu + (x-\mu)'\Sigma_\theta \Sigma_x^{-1} = \xi, \tag{6}$$

where

$$\xi_i = \theta_i^* - y_i^*\sqrt{\lambda_{ii}}, \quad i = 1,2,\ldots,k.$$

Where computer facilities are available, equidistributional contours may be drawn via the Newton-Raphson iteration process for nonlinear equations. For example, let $(y_1, y_2)'$ follow a standardized bivariate normal distribution with correlation ρ. Let α be any number between 0 and 1, and u be such that $\Pr(u \leq y_1) < \alpha$. We will search for the value v at which $G(v) = 0$, where

$$\begin{aligned} G(v) &= \Pr(y_1 \geq u, y_2 \geq v) - \alpha, \\ &= \Pr(y_1 \leq -u, y_2 \leq -v) - \alpha. \end{aligned} \tag{7}$$

The derivative of G(v) with respect to v is given as

$$G'(v) = -(2\pi)^{-\frac{1}{2}} \exp\left(-\frac{v^2}{2}\right) P(y_1 \leq -u \mid y_2 = -v). \tag{8}$$

Here the conditional distribution of y_1 given $y_2 = -v$ is a normal distribution with a mean of $-\rho v$ and a standard deviation of $(1-\rho^2)^{\frac{1}{2}}$. Hence

$$G'(v) = -(2\pi)^{-\frac{1}{2}} \exp\left(-\frac{v^2}{2}\right) P\left(Z \leq \frac{-u+\rho v}{(1-\rho^2)^{\frac{1}{2}}}\right) \tag{9}$$

where Z is the standardized normal variable. The values of G(v) and G'(v) may be obtained via computer programs such as MDBNOR (IMSL, 1977) and the Fortran IV library function ERFC. Both G(v) and G'(v) are needed in the Newton-Raphson iteration process. This procedure has been found to converge when u is not too close to the upper bound u_o at which $P(u_o \leq y_1) = \alpha$. (It may be noted that the bivariate equidistributional contour has two asymptotes defined as $u = u_o$ and $v = u_o$. Thus small variations in a u value near u_o will tend to associate with <u>substantial</u> changes in the v values; because of this, the iteration process may fail. However, since $P(y_1 \geq u, y_2 \geq v) = P(y_1 \geq v, y_2 \geq u)$, the contour is symmetric with respect to the first diagonal in the (u,v)-plane. Thus it is necessary to iterate the v value for each u sufficiently smaller than the upper bound u_o, and then to resort to symmetry to complete the drawing of the contour.)

The drawing of an equidistributional contour for any k-variate normal distribution may be accomplished in the same way via the Newton-Raphson iteration process previously described. The details are straightforward and therefore are not presented here. Multivariate normal probabilities of the form $P(y_1^* \leq y_1, y_2^* \leq y_2, \ldots, y_k^* \leq y_k)$ may be evaluated via computer programs such as the one described in Milton (1972).

It may be noted that the contour γ does not depend on the two vectors θ^* and μ. In addition, in the transformation from γ to C as defined by (6), these two vectors act only to indicate the new location of the transformed curve. It follows that the <u>shape</u> of the cutting contour C does not depend on either the vector μ or the vector θ^*.

5. AN ILLUSTRATION OF CUTTING CONTOUR

Consider now a selection based on two variables defined by the true scores θ_1 and θ_2, and by the observed test data x_1 and x_2. It will be assumed, as in Lord (1962), that both x_1 and x_2 are in their standardized form and have a common reliability coefficient of .90. In addition, let the correlation between w_1 and x_2 be .60. It follows that the matrices Σ_x and Σ_θ are defined as

$$\Sigma_x = \begin{pmatrix} 1.00 & .60 \\ .60 & 1.00 \end{pmatrix}$$

and

$$\Sigma_\theta = \begin{pmatrix} .90 & .60 \\ .60 & .90 \end{pmatrix}.$$

With

$$\Sigma_x^{-1} = \frac{1}{.64}\begin{pmatrix} 1.00 & -.60 \\ -.60 & 1.00 \end{pmatrix},$$

it follows that

$$\Sigma_\theta\Sigma_x^{-1} = \frac{1}{.64}\begin{pmatrix} .54 & .06 \\ .06 & .54 \end{pmatrix} = \begin{pmatrix} .84375 & .09375 \\ .09375 & .84375 \end{pmatrix}$$

and

$$\Lambda = \Sigma_\theta\Sigma_x^{-1}\Sigma_\theta = \frac{1}{.64}\begin{pmatrix} .522 & .378 \\ .378 & .522 \end{pmatrix} = \begin{pmatrix} .815625 & .590625 \\ .590625 & .815625 \end{pmatrix}.$$

Thus the posterior distribution of $\theta = (\theta_1,\theta_2)'$ given the test data $x = (x_1,x_2)'$ is bivariate normal with mean vector $\xi(x) = (\xi_1,\xi_2)'$ where $\xi_1 = .84375x_1 + .09375x_2$ and $\xi_2 = .09375x_1 + .84375x_2$. The posterior standard deviations are $(.815625)^{\frac{1}{2}} = .90312$ for both θ_1 and θ_2, and the posterior correlation between θ_1 and θ_2 is $.590625/.815625 = .72414$.

It may then be deduced from the equations represented by (4) that

$$y_1^* = (\theta_1^* - (.84375x_1 + .09375x_2))/.90312$$

and

$$y_2^* = (\theta_2^* - (.09375x_1 + .84375x_2))/.90312.$$

To draw the (x_1,x_2) contour line, let us suppose that $\theta_1^* = \theta_2^* = 0$. The two equations represented by (6) can be written as

$$.84375x_1 + .09375x_2 = -.90312y_1^*$$

$$.09375x_1 + .84375x_2 = -.90312y_2^*$$

or equivalently

$$x_1 = -1.08374y_1^* + .12042y_2^*$$

$$x_2 = .12042y_1^* - 1.08374y_2^*.$$

In the above equations, the point at coordinate (y_1^*,y_2^*) belongs to the equidistributional contour line defined by $P(y_1^* \leq y_1, y_2^* \leq y_2) = 1/(1+Q)$, where $(y_1,y_2)'$ follows a standardized bivariate normal distribution with correlation .72414. It may be recalled that Q is the ratio of the false negative error loss to the false positive error loss.

For purposes of illustration, the steps previously described were implemented in drawing the cutting contours associated with the loss ratios $Q = 1/3$, 1, and 4. These contours are depicted in Figure I.

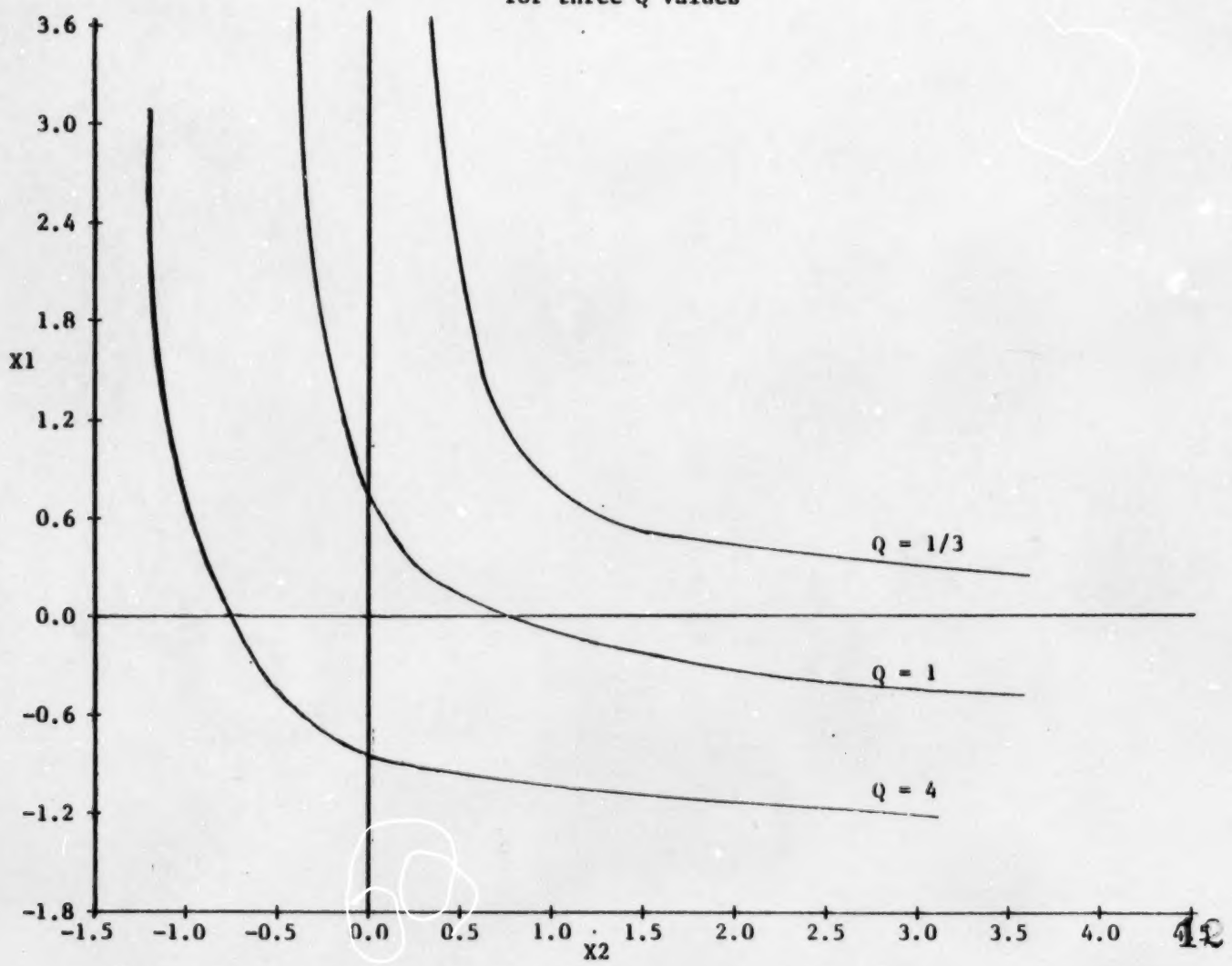
6. EFFECT OF LOSS RATIO ON CUTTING CONTOUR

In Figure I, the upper right region bounded by each cutting contour consists of the test score points at which mastery is granted. It may be observed that the mastery region expands as the loss ratio Q increases. This conclusion is to be expected. If the consequences due to a false negative error become more serious (i.e., Q increases), then the classification (or selection) procedure should be so designed as to reduce the probability of this error. Thus the size of the nonmastery set must be reduced, and as a consequence, it becomes more likely that mastery will be granted.

In general, let the set $A^*(Q_1)$ consist of all points $y^* = (y_1^*,y_2^*,\ldots,y_k^*)$ for which

$$P(y_1^* \leq y_1, y_2^* \leq y_2, \ldots, y_k^* \leq y_k) > 1/(1+Q_1) \qquad (10)$$

FIGURE I
Multivariate Cutting Contours
for three Q Values



and let $A(Q_1)$ be the corresponding region in the test score space. It may be verified that in $A(Q_1)$ the expected posterior losses associated with the two actions a_1 (granting mastery) and a_2 (denying mastery) satisfy the inequality $R(a_1|x) < R(a_2|x)$. Thus the set $A(Q_1)$ consists of test score points at which the subject is declared a master. Now let Q_2 be a second loss ratio such that $Q_1 < Q_2$. This is equivalent to $1/(1+Q_1) > 1/(1+Q_2)$. Let $A(Q_2)$ have the same meaning as above. Then any test score points which belong to $A(Q_1)$ must also belong to $A(Q_2)$. In other words, the inequality $Q_1 < Q_2$ implies that $A(Q_1) \subset A(Q_2)$. Thus, as the loss ratio Q increases, the mastery region in the test score space will expand. By the same line of reasoning, when Q decreases, the mastery region will be reduced in size.

7. EFFECT OF ERRORS OF MEASUREMENT ON CUTTING CONTOUR

To illustrate the effect of errors of measurement on the cutting contour in the test score space, let it be assumed as in the previous section that the test scores x_1 and x_2 are in their standardized forms and have a correlation of .60. In addition, let it be assumed that x_1 and x_2 are equally reliable with common reliability coefficient ρ, and that $\theta_1^* = \theta_2^* = 0$.

It follows from the equations represented by (6) that

$$\begin{aligned} 1.25(\rho-.36)x_1 + .75(1-\rho)x_2 &= -(\rho^2-.72\rho+.36)^{\frac{1}{2}}y_1^* \\ .75(1-\rho)x_1 + 1.25(\rho-.36)x_2 &= -(\rho^2-.72\rho+.36)^{\frac{1}{2}}y_2^*. \end{aligned} \tag{11}$$

In these expressions, the point (y_1^*, y_2^*) belongs to an appropriate equidistributional contour associated with the standardized bivariate normal distribution with correlation $\delta = .6(-\rho^2+2\rho-.36)/(\rho^2-.72\rho+.36)$.

It may be deduced from the positive semidefiniteness of the covariance matrix of (θ_1, θ_2) that the common reliability ρ must be between .60 and 1.00. As a function of ρ, the posterior correlation δ is a decreasing function, assuming the value of 1.00 when $\rho = .60$ and having the limit of .60 when ρ tends to 1.00.

When ρ approaches the upper limit 1.00, the posterior distribution of (θ_1, θ_2) will degenerate at the point (x_1, x_2). (It may be noted that when $\rho = 1$, the posterior covariance matrix Λ as defined in Section 4, i.e., $\Sigma_\theta - \Sigma_\theta \Sigma_x^{-1} \Sigma_\theta$, will vanish.) Given the test score vector $x = (x_1, x_2)'$, formally, the posterior expected loss for taking action a_1, $R(a_1|x)$, is equal to 0 when $x \in \Omega$ and 1 when $x \in \Omega^c$. Similarly, $R(a_2|x)$ is equal to Q when $x \in \Omega$ and 0 when $x \in \Omega^c$. Thus, mastery is granted when $x_1 \geq 0$ and $x_2 \geq 0$. When either $x_1 < 0$ or $x_2 < 0$ (or both), mastery is denied. In summary, when ρ tends to unity, the cutting contour line in the test score space will approach the cutting contour line defined in the true score space.

Consider now the other limiting situation where ρ tends to .60 and δ goes to 1.00. The entire bivariate probability of $(x_1, x_2)'$ is now concentrated on the diagonal $x_1 = x_2$. Let y_o be the point at which $P(y_o \leq y_1) = 1/(1+Q)$ where y_1, as previously defined, is a standardized normal variable. The equidistributional contour line is now comprised of the two half lines defined by (i) $y_1^* = y_o$ and $y_2^* \leq y_o$, and (ii) $y_2^* = y_o$ and $y_1^* \leq y_o$. Both half lines start at the point (y_o, y_o) and extend to $-\infty$, one vertically and the other horizontally.

The equations (11) now become

$$x_1 + x_2 = -1.7889 y_1^*$$

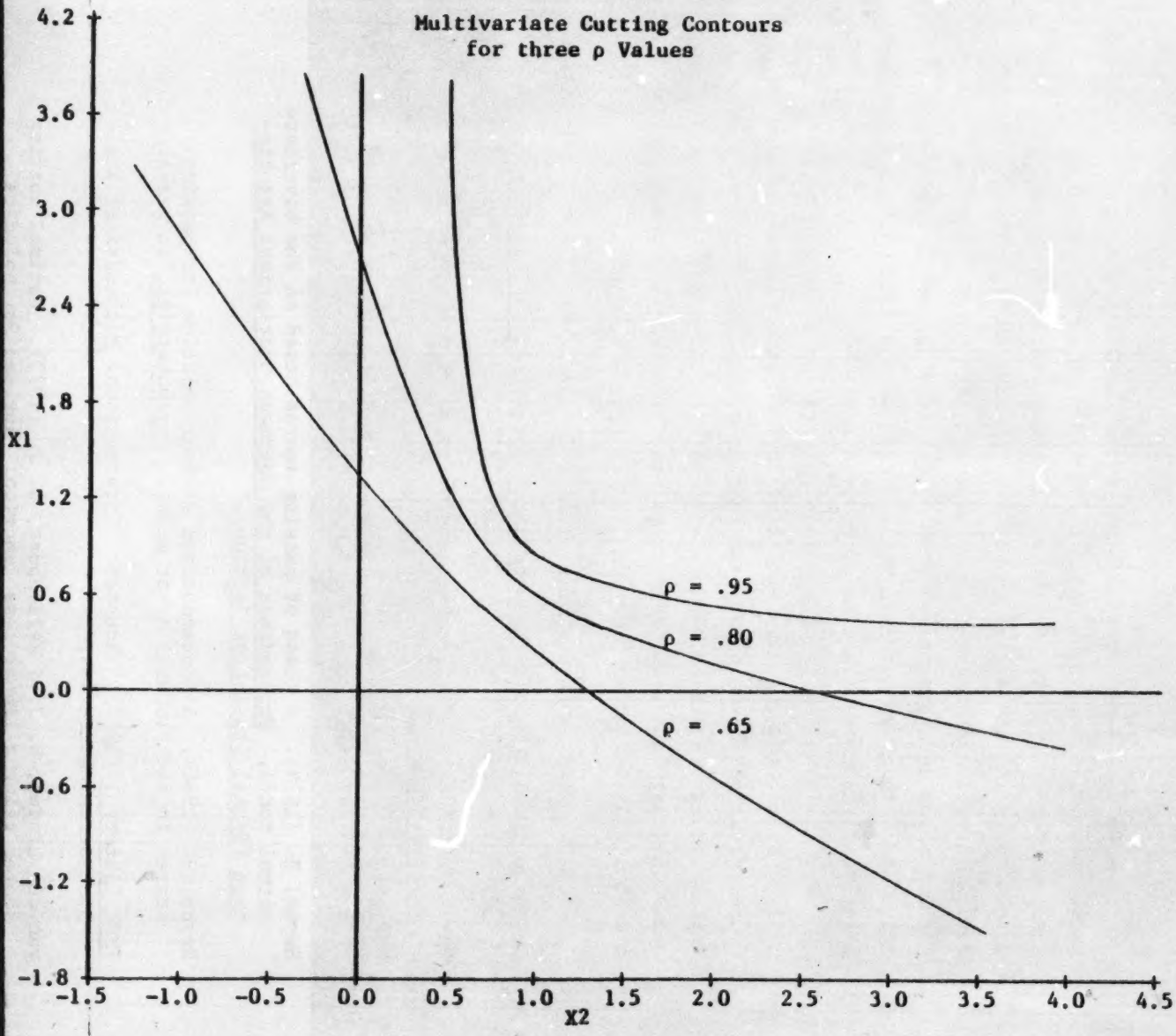
$$x_1 + x_2 = -1.7889 y_2^*.$$

It follows that the cutting contour in the observed test score space is the straight line defined by the equation $x_1 + x_2 = -1.7889 y_o$. The decision regarding granting or denying mastery in this case is actually based on the composite score $x_1 + x_2$ although <u>separate</u> cutting scores have been set in the true score space!

For purposes of illustration, cutting contours are drawn for the reliability coefficients of $\rho = .95$, .80, and .65. The contours are shown in Figure II.

FIGURE II

Multivariate Cutting Contours
for three ρ Values



8. SUMMARY

A general framework for making mastery/nonmastery decisions based on multivariate test data is described in this study. Over all, mastery is granted (or denied) if the posterior expected loss associated with such action is smaller than the one incurred by the denial (or grant) of mastery. An explicit form for the cutting contour which separates mastery and nonmastery states in the test score space is given for multivariate test scores which follow a normal distribution with a constant loss ratio.

For the case involving multiple cutting scores in the true ability space, the test score cutting contour will resemble the boundary defined by multiple test cutting scores when the test reliabilities are reasonably close to unity. For tests with low reliabilities, decisions may very well be based simply on a suitably chosen composite score.

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